QUESTIONS

a) Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given parametric equations $x = \sin(t)$ and $y = 1 + \tan t$. Give your answers only in terms of sec and tan.

b) Use logarithmic differentiation to find the derivative of the function $y = \frac{x^2\sqrt[4]{(x-1)^3}}{(x+2)^2}$ with domain of $x > 1$

c) What is the equation of the tangent line and normal line of the curve $xy^3 + 2x^2y - x^2 - y = 1$ at the point (1, 1)?

d) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x^2 - 3xy + 2y^2 = 4$. List you answer in the format of a single fraction.

e) Given $y = 4x - 19\ln x - \frac{12}{x}$, find the coordinates for any critical point/s and inflexion point/s that exist for the function. Clearly classify each type of point that has been found.

ANSWERS

a. The first derivative of parametric equations is obtained from the quotient of the derivative of y with respect to t and the derivative of x with respect to t.

$$\frac{dy}{dt} = d[1 + \tan(t)] = sec^2(t)$$

$$\frac{dx}{dt} = d[\sin(t)] = cos(t) = \frac{1}{\sec(t)}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{sec^2(t)}{\frac{1}{\sec(t)}} = \boldsymbol{sec^3(t)}$$

The second derivative of parametric equations is taken from the derivative of its first derivative. Note that chain rule should be followed when taking derivatives of functions.

$$\frac{d^2y}{dx^2} = d\left(\frac{dy}{dx}\right) = d[sec^3(t)] = 3sec^2(t)[\sec(t)\tan(t)] = \mathbf{3}\boldsymbol{sec^3(t)}\mathbf{tan}\,\boldsymbol{(t)}$$

b. Logarithmic differentiation is done be first taking the natural logarithm of both sides of the equations and then getting the derivative of each term. Note that the derivative of ln[f(x)] is equal to [1/f(x)][f'(x)].

$$lny = ln\left(\frac{x^2\sqrt[4]{(x-1)^3}}{(x+2)^2}\right)$$

$$lny = \ln(x^2) + \ln(x-1)^{\frac{3}{4}} - ln(x+2)^2$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2} + \frac{\frac{3}{4}(x-1)^{-\frac{1}{4}}}{(x-1)^{\frac{3}{4}}} - \frac{2(x+2)}{(x+2)^2}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{3}{4(x-1)} - \frac{2}{x+2}\right) = \boldsymbol{\left(\frac{x^2\sqrt[4]{(x-1)^3}}{(x+2)^2}\right)\left(\frac{2}{x} + \frac{3}{4(x-1)} - \frac{2}{x+2}\right)}$$

c. The equation of the tangent line to a curve is determined by the first derivative or slope at a particular point in the curve. The equation of the normal line to a curve has a slope that is the negative reciprocal of the slope of the tangent line. To determine the equations of these lines, obtain first an equation for the derivative of the curve and then substitute the given point to the equation to get the slope of the tangent line.

$$d[xy^3 + 2x^2y - x^2 - y] = d[1]$$

$$y^3 + 3y^2x\frac{dy}{dx} + 2x^2\frac{dy}{dx} + 4xy - 2x - \frac{dy}{dx} = 0$$

$$(1)^3 + 3(1)^2(1)\frac{dy}{dx} + 2(1)^2\frac{dy}{dx} + 4(1)(1) - 2(1) - \frac{dy}{dx} = 0$$

$$1 + 3\frac{dy}{dx} + 2\frac{dy}{dx} + 4 - 2 - \frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} + 3 = 0$$

$$\frac{dy}{dx} = -\frac{3}{4}$$

Use the point-slope form of a line to setup the equation of the tangent line. For the normal line, use the same form of a line but change the slope to its negative reciprocal.

Tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{3}{4}(x - 1)$$

$$\boldsymbol{y = -\frac{3}{4}x + \frac{7}{4}}$$

Normal line:

$$y - 1 = \frac{4}{3}(x - 1)$$

$$\boldsymbol{y = \frac{4}{3}x - \frac{1}{3}}$$

d. In implicit differentiation, the derivative of both sides of the equation with two variables x and y is obtained by treating one of the variables constant while taking the derivative of the other for each term in the equation.

$$d[x^2 - 3xy + 2y^2] = d[4]$$

$$2x - 3x\frac{dy}{dx} - 3y + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4y - 3x) = 3y - 2x$$

$$\boldsymbol{\frac{dy}{dx} = \frac{3y - 2x}{4y - 3x}}$$

To get the second derivative, take the derivative of the first derivative using the quotient rule: $D = \frac{vdu - udv}{v^2}$. Substitute the equation for dy/dx to the resulting equation.

$$d\left[\frac{dy}{dx}\right] = d\left[\frac{3y - 2x}{4y - 3x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(4y - 3x)\left(3\frac{dy}{dx} - 2\right) - (3y - 2x)\left(4\frac{dy}{dx} - 3\right)}{(4y - 3x)^2}$$

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{(4y - 3x)\left(3 \cdot \frac{3y - 2x}{4y - 3x} - 2\right) - (3y - 2x)\left(4 \cdot \frac{3y - 2x}{4y - 3x} - 3\right)}{(4y - 3x)^2}}$$

e. Critical points occur when the first derivative of a function is 0 or undefined. On the other hand, inflection points occur when the second derivative is 0 or undefined. To determine these points, obtain first the first and second derivatives of the given function and then equate them to 0 then solve for the values of x that will make the equation true. Alternatively, think of values of x that will result to undefined derivatives.

$$d[y] = d\left[4x - 19lnx - \frac{12}{x}\right]$$

$$\frac{dy}{dx} = 4 - \frac{19}{x} + \frac{12}{x^2}$$

Notice that when x = 0, the first derivative is undefined. Therefore, one of the critical points is **x = 0**.

Equating the first derivative to 0 and multiplying by x^2:

$$x^2\left(4 - \frac{19}{x} + \frac{12}{x^2} = 0\right)$$

$$4x^2 - 19x + 12 = 0$$

Factoring to get the other critical points:

$$(4x - 3)(x - 4) = 0$$

$$\boldsymbol{x = 4} \qquad \boldsymbol{x = \frac{3}{4}}$$

Taking the derivative of the first derivative to get the second derivative:

$$d\left[\frac{dy}{dx}\right] = d\left[4 - \frac{19}{x} + \frac{12}{x^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{19}{x^2} - \frac{24}{x^3}$$

Notice that when x = 0, the second derivative is undefined. Thus, one of the inflection points is **x = 0**.

Equating the second derivative to 0 and multiplying by x^3:

$$x^3\left(\frac{19}{x^2} - \frac{24}{x^3} = 0\right)$$

Simplifying to get the other inflection point:

$$19x - 24 = 0 \qquad \boldsymbol{x = \frac{24}{19}}$$